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Self-similar asymptotics of wave problems and the structures of non-classical discontinuities in non-linearly elastic media with dispersion and dissipation☆

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Abstract

Solutions of the non-linear hyperbolic equations describing quasi-transverse waves in composite elastic media are investigated within the framework of a previously proposed model, which takes into account small dissipative and dispersion processes. It is well known for this model that if a solution of the problem of the decay of an arbitrary discontinuity is constructed using Riemann waves and discontinuities having a structure, the solution turns out to be non-unique. In order to study the problem of non-uniqueness, solutions of non-self-similar problems are constructed numerically within the framework of the proposed model with initial data in the form of a "smooth" step. With time passing the solutions acquire a self-similar asymptotic form, corresponding to a certain solution of the problem of the decay of an arbitrary discontinuity. It is shown that, by changing the method of smoothing the step, one can construct any of the self-similar asymptotic forms, as was done previously in Ref. [Chugainova AP. The asymptotic behaviour of non-linear waves in elastic media with dispersion and dissipation. *Teor Mat Fiz* 2006;**147**(2):240–56] for media with terms of opposite sign, responsible for the non-linearity, although the set of admissible discontinuities and the structure of the solutions of the problems in these cases turn out to be different.

Long-wave perturbations in elastic media are described by non-linear hyperbolic equations, which express the conservation laws. Discontinuities arise in the solutions of these equations in the process of evolution. A discontinuity in the case considered is classical (or a shockwave)[2] if the relations which are satisfied on the discontinuity follow from the conservation laws and the evolution conditions are satisfied. If the discontinuity possesses a structure, within the framework of the model employed, it is said to be admissible. The discontinuity will be non-classical (or specific) if, in addition to the conservation laws on the front, additional relations are satisfied which ensure the existence of a structure of the front and, in particular, determine its propagation velocity. Additional relations and, consequently, the behaviour of discontinuities and solutions involving the discontinuities, depend on the physical processes inside the structure. In this case, depending on the model of the fine-scale processes used, not all the classical discontinuities may turn out to be admissible.

The effect of dispersion (in addition to the existing dissipation) gives rise to a situation where the set of admissible discontinuities acquires a complex structure. If, as the model of the large-scale phenomena, we use a hyperbolic system

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of equations, which express the conservation laws, and a system of relations on the discontinuities, associated with it, with additional requirements that the discontinuities belong to a set of admissible discontinuities, then, as shown in Refs. 1, 3–5, when investigating non-linear waves in rods and in magnetic materials, and also for a certain model of a composite medium, the solutions of self-similar problems for defined regions of the initial parameters, turn out to be non-unique. The number of solutions in these regions depends on the effect of the dispersion inside the structure of the discontinuities (when the viscosity does not change) and increases without limit as this effect increases. Below we investigate numerically self-similar asymptotic forms in regions of non-uniqueness as a result of the evolution of non-stationary solutions of the complete system of partial differential equations.

1. A model of large-scale phenomena

Slightly non-linear quasi-transverse waves, propagating on a uniform background in the positive direction of the x axis, when the medium is slightly anisotropic, can be described by a simplified system of equations, which follows from the system of equations of the non-linear theory of elasticity[2,6]

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x}\frac{\partial R(u_1, u_2)}{\partial u_\alpha} = 0, \quad \alpha = 1, 2$$
$$u_\alpha = \frac{\partial w_\alpha}{\partial x} = u_\alpha(x, t), \quad R(u_1, u_2) = f\frac{u_1^2 + u_2^2}{2} + g\frac{u_2^2 - u_1^2}{2} - \kappa\left(\frac{u_1^2 + u_2^2}{2}\right)^2, \ f, g, \kappa = \text{const} \tag{1.1}$$

Here w_α are the displacements of the particles, considered as functions of the Lagrange coordinates $x_1, x_2, x_3 = x$, $g > 0$ is the anisotropy parameter, κ is a constant with the dimension of velocity, which characterizes non-linear effects, and f is the characteristic velocity when there is no non-linearity and no anisotropy (i.e. when $\kappa = 0$ and $g = 0$). The sign of the elastic constant κ has an essential effect on the behaviour of quasi-transverse simple and shock waves; in this paper we will assume that $\kappa > 0$.

We will consider self-similar solutions of system (1.1) of the form $u_\alpha = u_\alpha(x/t)$. These are Riemann waves and shock waves.

1.1. Riemann waves

Hyperbolic system (1.1) has two families of characteristics – fast and slow with velocities c_1 and c_2 respectively, $c_1 \le c_2$. Solutions of system (1.1) of the form $u_\alpha = u_\alpha(\theta(x, t))$ were investigated in Refs. 6 and 7, where θ is an arbitrary function of x and t, and expressions were obtained for the characteristic velocities c_α. These solutions are called Riemann waves. In the case of self-similarity $\theta = x/t$. In Fig. 1 we show, in the (u_1, u_2) plane, two orthogonal families of integral curves, corresponding to changes in the values of u_α in the fast and slow Riemann waves. In the case when $\kappa > 0$ considered here, the continuous curves correspond to fast waves while the dashed curves correspond to slow waves. The arrows in Fig. 1 indicate the directions in which c_α decreases along the integral curves. In self-similar Riemann

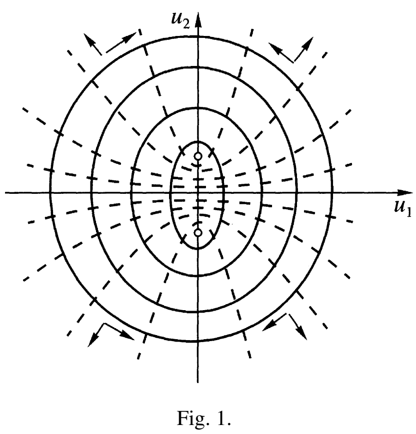

Fig. 1.

waves motion along the integral curve in the direction in which the characteristic velocity decreases corresponds to a change in the values of u_α as t increases when $x = \text{const}$.

Shock waves. The relations on the discontinuity[6,8]

$$[\partial R/\partial u_\alpha] - W[u_\alpha] = 0, \quad \alpha = 1, 2 \tag{1.2}$$

correspond to the system of Eq. (1.1), where $W = dx/dt$ is the Lagrange velocity of the discontinuity. Jumps in the corresponding quantities on the discontinuity front are denoted by square brackets: $[v] = v^l - v^r$, v^l is the value behind the discontinuity, and v^r is the value in front of the discontinuity. Henceforth the superscripts l and r refer to the parameters behind and in front of the discontinuity.

Eliminating W, we can write the equation of the shock adiabatics

$$(u_1^2 + u_2^2 - U_1^2 - U_2^2)(U_1 u_2 - U_2 u_1) + 2g\kappa^{-1}(u_1 - U_1)(u_2 - U_2) = 0$$

where U_1 and U_2 are the values of u_1 and u_2 in front of the discontinuity. Quantities behind the discontinuity satisfy the equation of the shock adiabatic and are denoted, as previously, by u_1 and u_2.

In Fig. 2a,b we show the shock adiabatics in the (u_1, u_2) plane (the point $A(U_1, U_2)$ corresponds to the state in front of the discontinuity) for values of the parameters $U_1 = 2$, $U_2 = 0.4$, $g = 1$ and $\kappa = 4$ (Fig. 2,a) and $U_1 = 1$, $U_2 = 0.1$, $g = 1$ and $\kappa = 2$ (Fig. 2,b).

Not all the states u_1 and u_2, which belong to the shock adiabatic, may be states behind the jump front. If the relations on the discontinuities are represented by system (1.2), the evolution requirements have the form[2,6]

$$c_2^r \le W, \quad c_1^l \le W \le c_2^l \tag{1.3}$$

$$c_1^r \le W \le c_2^r, \quad W \le c_1^l \tag{1.4}$$

Relations (1.3) and (1.4) respectively define evolution fast and slow shock waves. For a clear representation of inequalities (1.3) and (1.4) it is more convenient to use an evolution diagram (Fig. 2,a,b), where, along mutually orthogonal axes, we plot the velocities involved in the inequalities, and we thereby distinguish in the plane the regions where each of the inequalities is satisfied.[6] For a fixed state ahead of the front along the horizontal axis of the evolution diagram, the

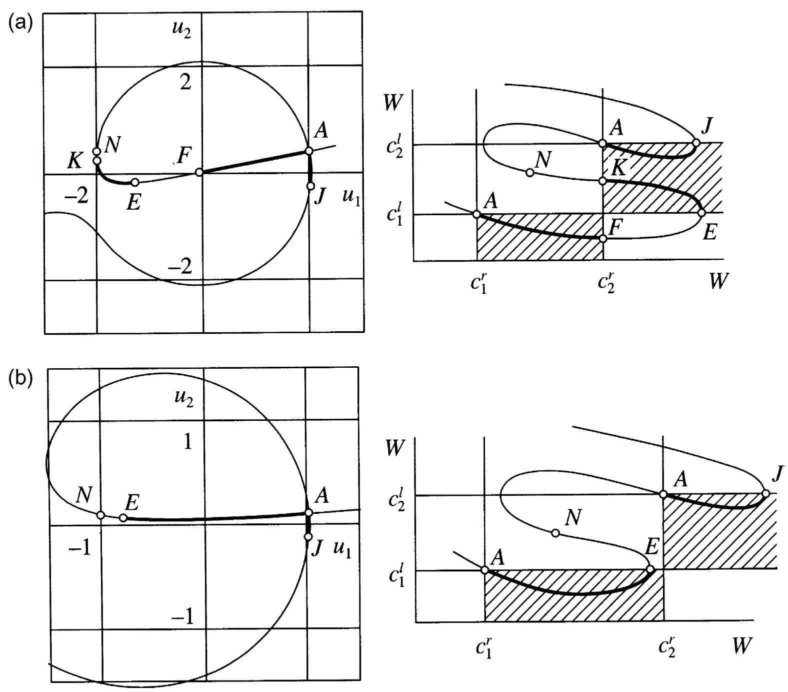

Fig. 2.

velocity of the discontinuity and also the velocities of the characteristics are reproduced in the same scale. Along the vertical axis we have only retained the inequalities between W, c_1^l and c_2^l. Regions where both inequalities of systems (1.3) and (1.4) are satisfied form the hatched rectangles in the plane of Fig. 2,a,b.

The relations on the discontinluity (1.2) enable us to obtain the velocity W and other quantities along the shock adiabatic and to represent the shock adiabatic in this plane. Here the values of W are correctly represented along the horizontal axis and only qualitatively along the vertical axis. In the evolution diagrams (Fig. 2) the initial point A is represented by two points, by virtue of its positions on two intersecting branches of the shock adiabatic. The sections of the shock adiabatic which fall into the evolution rectangles correspond to the evolution fast shock waves in the upper evolution rectangle and correspond to slow shock waves in the lower evolution rectangle. These sections on the shock adiabatic in the (u_1, u_2) plane in Fig. 2,a,b are shown by the heavy continuous segments. In Fig. 2,a,b the same letters refer to the same points. The situation shown in Fig. 2,a, corresponds to states in front of the discontinuity with relatively large U_1 and U_2 (such that $\kappa(U_1^2 + U_2^2 > 4g)$). In this case there are three evolution sections: sections AJ and KE correspond to possible states behind the fast evolution shock waves, and the section AF corresponds to possible states behind the slow shock waves. For fairly small U_1 and U_2 the shock adiabatic and the evolution diagram have the form shown in Fig. 2,b. In this case there are two evolution sections. The boundary separating case 2,a and case 2,b was investigated in Ref. 6 in the U_1, U_2 plane.

In Fig. 2 the points A, J, E and K are called Jouguet points, and at these points the velocity of the shock wave is identical with one of the characteristic velocities in front of or behind the jump. The Jouguet points are boundary points of the evolution intervals.

In all parts of Fig. 2 we have shown the characteristic point N, and the point A symmetrical with respect to the u_2 axis. At the corresponding jump $A \to N$, not satisfying inequalities (1.3) and (1.4), the quantity u_1 changes sign while the quantity u_2 remains unchanged.

In the case considered, all the evolution discontinuities satisfy the requirement of non-decreasing entropy.[6]

An investigation of the structure of the discontinuities will serve as a stricter selection rule. The formulation of this problem requires an improvement of the model of the medium and is therefore outside the framework of the classical theory of elasticity. We will consider this problem in the next section.

2. Non-linear quasi-transverse waves in composite media

In what follows we will investigate the solutions of the system of equations describing the behaviour of non-linear waves in elastic composite materials, i.e., in an elastic medium which possesses an internal structure, associated with the periodical of the properties of a medium

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x}\frac{\partial R(u_1, u_2)}{\partial u_\alpha} = \mu\frac{\partial^2 u_\alpha}{\partial x^2} + (-1)^{3-\alpha} m \frac{\partial^2 u_{3-\alpha}}{\partial x^2}, \quad \alpha = 1, 2; \; \mu, m = \text{const}, \; \mu \geq 0 \tag{2.1}$$

System (2.1) differs from (1.1) in that it takes into account dissipative and dispersion effects which are important when describing the structure of discontinuities. Terms with the factor m describe dispersion while those with the factor μ describe dissipation. It was assumed that for small non-linearity, the terms describing the dispersion and dissipation retain the same form as in linear waves. The form of the dispersion terms in system (2.1) corresponds to one of the versions of the structure of composites, the behaviour of which was investigated in Ref. 9.

The factors m and μ accompany the leading derivatives in system (2.1), and hence when investigating large-scale phenomena this system will be identical with system (1.1). The form of the function $R(u_1, u_2)$ is assumed to be the same as in system (1.1).

The structure of discontinuities. For system of Eq. (2.1) we will investigate solutions of the form

$$u_\alpha = u_\alpha(\xi), \quad \xi = -x + Wt$$

(W=const is the velocity of the discontinuity) such that when $\xi \to -\infty$ the quantities u_α approach the values of U_1 and U_2, corresponding to the state in front of the discontinuity, while when $\xi \to +\infty$ they take values of u_1 and u_2 corresponding to the state behind the discontinuity, i.e. the state which belongs to a certain point of the shock adiabatic.

When $\mu \to 0$, $m \to 0$ and $m/\mu = \text{const}$ these solutions become discontinuities, the change in the values of which is identical with the change in the values in the corresponding quasi-transverse discontinuity.

For the functions $u_\alpha(\xi)$ we have a system of two second-order ordinary differential equations. Integrating once with respect to ξ, we obtain the system

$$\mu\frac{du_\alpha}{d\xi} + (-1)^{3-\alpha}m\frac{du_{3-\alpha}}{d\xi} = -\frac{\partial Z}{\partial u_\alpha}, \quad \alpha = 1, 2$$

$$Z(u_1, u_2) = -R(u_1, u_2) + (2)^{-1}W(u_1^2 + u_2^2) + Q_1u_1 + Q_2u_2 \tag{2.2}$$

$$Q_\alpha = U_\alpha(f - W + (-1)^\alpha g - \kappa(U_1^2 + U_2^2))$$

Note that

$$\frac{dZ}{d\xi} = -\mu\left[\left(\frac{du_1}{d\xi}\right)^2 + \left(\frac{du_2}{d\xi}\right)^2\right] < 0$$

System (2.2) is identical, apart from the notation, with the system describing the structure of non-linear electromagnetic waves in slightly anisotropic magnetic materials.[2,5]

When investigating non-linear wave processes, when, in describing the structure of discontinuities, dispersion and dissipation processes are important, it was shown in Refs. 3 and 5 that the set of admissible values of u_α behind the discontinuity depends essentially on the value of the ratio m/μ.

The coordinates of the singular points of system (2.2) are defined by the system of equations

$$\frac{\partial Z}{\partial u_1} = 0, \quad \frac{\partial Z}{\partial u_2} = 0$$

We emphasise that in the case when $m/\mu \gg 1$ the integral curves of system (2.2) are close to level lines $Z(u_1, u_2) = \text{const}$. In Fig. 3 we show the level lines for values of the parameters $f = 1$, $g = 1$, $\kappa = 4$ ($W = -16$ and the states in front of the discontinuity $U_1 = 2$ and $U_2 = 0.4$ (point A)).

The range of the velocity variation

$$W_N < W < \min(c_2^r, W_E)$$

is of most interest (W_N and W_E are the velocities of the discontinuities $A \to M$ and $A \to E$ respectively, see the evolution diagrams in Fig. 2), since in this range there are five singular points in system (2.2): the initial point A – a saddle point, the stable focus O (the minimum of the function $Z(u_1, u_2)$), the saddle C and two unstable foci B and D (the maxima of the function $Z(u_1, u_2)$). When $W > c_2^r$ and $W < W_N$ there are no specific discontinuities. Note that when $W > c_2^r$ the singular point A in the (u_1, u_2) plane is a node or a focus with departing integral curves. In this case there are no

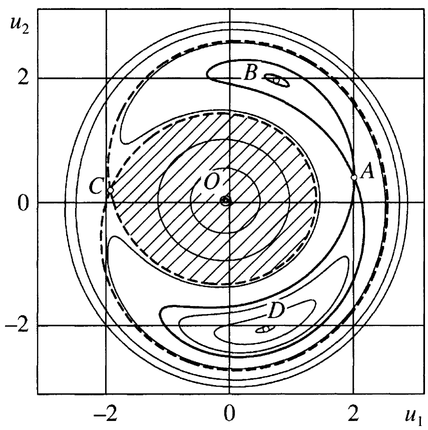

Fig. 3.

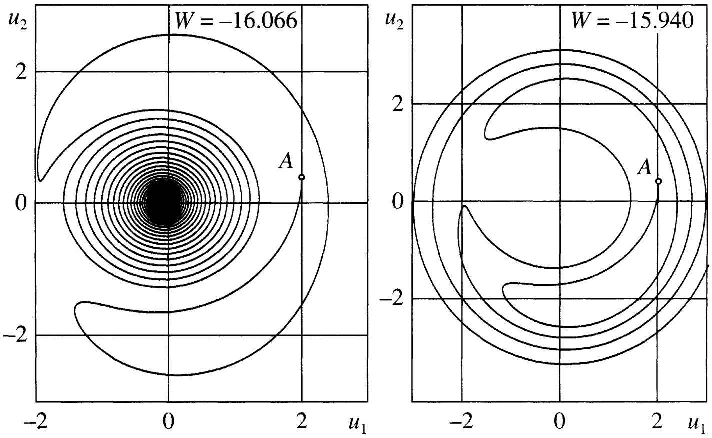

Fig. 4.

specific discontinuities, since for these to exist it is necessary that the point A should be a saddle point. Each of the two integral curves, emerging from the point A, when $\xi \to +\infty$ must arrive at one of the singular points O, C or go to infinity. Examples of the two characteristic stationary structures are shown in Fig. 4,a,b. The arrival of an integral curve at the saddle point C must be regarded as an exceptional case, which is possible for a special discrete set of the velocity values $W = W_i^*$.

The integral curves emerging from the point A move towards a reduction in the function Z, intersecting the level line Z = const at small angles (of the order of μ/m). Before any of the two integral curves intersects the dashed level line $Z = Z(C)$ (Fig. 3), for fairly small values of μ/m it makes many rotations around the heavy continuous level line ($Z = Z(A)$). When W changes, the point of intersection of the integral curve and the dashed level line moves along the dashed level line. When the above-mentioned number of rotations changes by unity, the point of intersection traverses the whole dashed level line, twice being at the point C. For appropriate values of W a structure of the specific discontinuity $A \to C$ exists. If the point of intersection of the integral curve and the level line is situated on part of the dashed level line which is the boundary of the hatched region, this integral curve further arrives at the point O, representing the structure of the slow shock wave $A \to O$. If the integral curve intersects part of the dashed level line, which is not the boundary of the hatched region, it does not correspond to the structure of any discontinuity.

In Figs. 5 and 6 we show examples of stationary structures of specific discontinuities, obtained numerically for values of the dissipation parameter $\mu = 0.05$ and the dispersion $m = 3$ ($m/\mu = 60$) and the previous values $U_1 = 2$, $U_2 = 0.4$, $f = 1$,

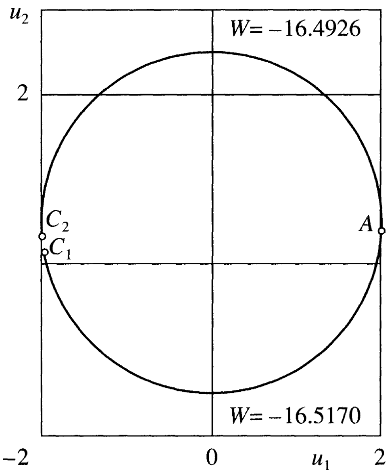

Fig. 5.

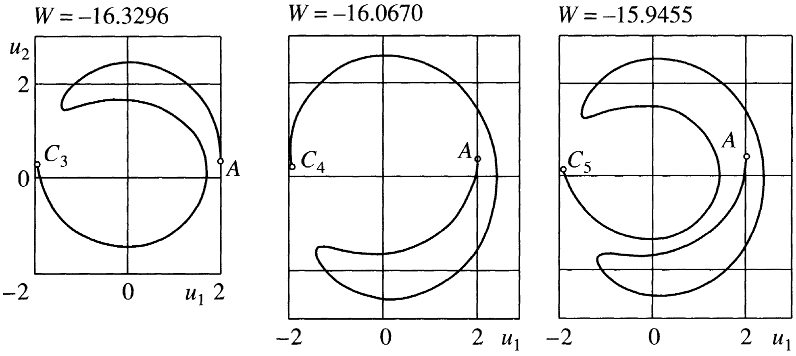

Fig. 6.

$g = 1$ and $\kappa = 4$. In Fig. 5 we show simple structures of specific discontinuities and in Fig. 6 we show complex structures for large values of $W = W^*$, which demonstrate the different forms of bypassing the heavy level line in Fig. 3.

For the chosen ratio $m/\mu = 60$, as a result of a numerical calculation we obtain a set of values of $W_i^*(i = 1, 2, \ldots, 5)$, for which one of the two possible integral curves, emerging from the initial saddle point A in opposite directions, arrives at the saddle point $C_i(i = 1, 2, \ldots, 5)$. The points C_i correspond to specific discontinuities (Fig. 7). Each value of W_i^* corresponds to a certain set of stationary points of system (2.2), which distinguish the evolution sections on the shock adiabatic. In Fig. 7,a,b we show fragments of the shock adiabatic and diagrams of the evolution for the same parameters f, g, κ and U_α, as in the curves shown in Fig. 2 and described in Section 1. In Fig. 7, on fragments of the shock adiabatic and on the diagrams of the evolution, we show possible states behind the permissible discontinuities for the initial state A; these are shown by the heavy lines and the dark dots. The evolution sections AF (Fig. 7,a) and AE (Fig. 7,b) of the shock adiabatic are divided into sections, where some of them do not correspond to admissible discontinuities. On the non-evolution section KN (Fig. 7,a) and the non-evolution section EN (Fig. 7,b) we show the points C_i, which correspond to evolution specific discontinuities.

When the ratio m/μ increases the number of separate points and sections of the shock adiabatic, corresponding to the permissible discontinuities, increases.

In the case considered above, the coordinates of the point A are chosen so that the point is situated close to the u_1 axis. An investigation of the pattern of integral curves when the coordinate u_2 of the point A increases showed that the presence of separate points C_i is possible when the ratio μ/m is reduced considerably.

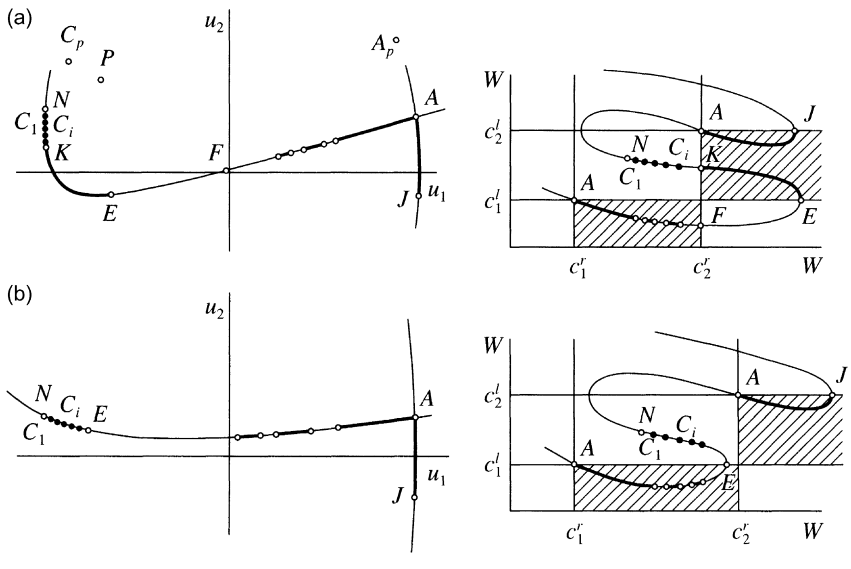

Fig. 7.

Note that when $m=0$ all the evolution discontinuities are permissible (see Fig. 2).[6]

The non-uniqueness of the solutions of the self-similar wave problem. An investigation of the structure of the discontinuities will make the large-scale model for determining the set of permissible discontinuities, formulated in Section 1, more specific. We will show that the large-scale model does not guarantee uniqueness of the solutions. Non-uniqueness arises due to the presence on the shock adiabatic of a set of isolated points with separate values of the velocity of the discontinuity W_i^*. These points lie inside the rectangle $c_1^r \le W \le c_2^r$, $c_1^l \le W \le c_2^l$ on the evolution diagram (Fig. 7,a,b) and correspond to specific discontinuities.

Within the framework of the model considered, which describes non-linear waves in composite media, for the class of initial conditions the solution of the self-similar wave problem is identical with the solution of the similar problem constructed using classical evolution discontinuities (the detailed pattern of these solutions was obtained previously in Ref. 10). Nevertheless, it must be emphasised that in certain regions of the initial parameters, a solution within the framework of the model considered here can only be constructed using admissible discontinuities and is not identical with the solution obtained earlier.

We will consider the example of the construction of a solution of the self-similar wave problem using specific discontinuities. The problem of the decay of an arbitrary discontinuity for system of Eq. (2.1) is formulated as follows: waves propagate in the region $x>0$, and the initial conditions ($t=0$) when $x>a$ correspond to the coordinates of the point A in the (u_1, u_2) plane (Fig. 7,a), when $0<x<a$ are the coordinates of the point P. Riemann waves or shock waves of moderate amplitude can propagate in front of or behind a specific discontinuity. Fast waves propagate in front of a specific discontinuity and slow waves propagate behind it.

One of the solutions of the problem is the following sequence of waves: a fast simple wave $A \to A_p$ propagates in front (Fig. 7,a), which transfers the state A to the point A_p such that the state behind the specific discontinuity $A_p \to C_p$ is characterized by the point C_p, representing the state in front of the slow shock wave $C_p \to P$. A slow shock wave $C_p \to P$ propagates behind the specific discontinuity $A_p \to C_p$. It is obvious that such solutions occur as many times as there are separate points C_i on the shock adiabatic, corresponding to the initial point A (there are five such points in the case considered.

Note that, unlike discontinuities which satisfy the evolution conditions, following from the conservation laws, specific discontinuities may follow one another since, depending on the state behind the specific discontinuity another specific discontinuity can propagate with a slower velocity, and hence, together with the solution described above there can also be other solutions involving sequences of specific discontinuities.

A qualitative analysis of the solutions shows that, for a given initial state on the right and various states on the left of the discontinuity there is a finite region of boundary values in the (u_1, u_2) plane, for which the solution of the self-similar problem is non-unique. The initial sets of admissible specific discontinuities leads to non-uniqueness of the solutions of the problem. A more detailed investigation of the solutions and of the problem of choosing solutions in the region of non-uniqueness is carried out below using the numerical solution of the generalized problem of the decay of an arbitrary discontinuity for system of Eq. (2.1).

The construction of self-similar asymptotic forms in the non-uniqueness region as the limit of non-stationary solutions of a system of partial differential equations. The generalized problem of the decay of an arbitrary discontinuity is formulated as follows: for system (2.1) the initial conditions at $t=0$ are specified in the form

$$u_\alpha = \begin{cases} u_\alpha^l & \text{When} \quad x < a \\ u_\alpha^0(x) & \text{When} \quad a \le x \le b, \quad \alpha = 1, 2; \quad a, b, u_\alpha^l, u_\alpha^r = \text{const} \\ u_\alpha^r & \text{When} \quad b < x \end{cases} \tag{2.3}$$

The functions $u_\alpha^0(x)$ determine the solution of the problem of the decay of an arbitrary discontinuity, including, as will be shown below, one or other asymptotic form as $t \to \infty$.

Below we will present the results of a numerical solution of a number of initial-boundary-value problems for system (2.1), for which the self-similar solutions described above may represent the asymptotic forms as $t \to \infty$. System (2.1) was written in the form of implicit non-linear difference equations, to which Newton's method was initially applied, followed by the matrix double-sweep method.[11] The calculation was carried out in the region $t \ge 0$, $0 \le x \le x^r$ with fixed right and left boundaries.

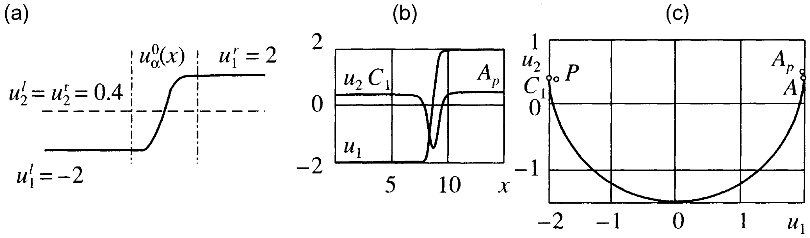

Fig. 8.

The waves involved in the solution possess different propagation velocities, and hence, as the time t between individual perturbations increases, sections will occur the length of which increases as t increases, corresponding to constant values of u_1 and u_2, while the structures of the shock waves will tend to stationary structures as $t \to \infty$. In view of this, to identify the waves it is necessary to obtain a solution for fairly long times. In this connection it is necessary to take a large section of the x axis, in order that the effect of the boundaries should not distort the solution.

We investigated the stability of the stationary structures numerically and, in particular, the structures of the specific discontinuities with respect to various one-dimensional perturbations. For this purpose we solve the following initial-boundary-value problems for system (2.1): the right boundary conditions u^r_α (for $x = x^r$, $t \geq 0$) correspond to the coordinates of the initial point A, while the left boundary conditions u^1_α (for $x = 0$, $t \geq 0$) correspond to the coordinates of point C_i; we specified different perturbations of the functions $u^0_\alpha(x)$ as the initial condition, where the functions $u^0_\alpha(x)$ were the solution of system (2.2) for given values of u^r_α and of the velocity W^*, corresponding to the chosen point C_i. Perturbations $\Delta u_\alpha(x)$ were specified in front of the stationary structure being investigated. Depending on the amplitude and width of the perturbation, a solution in the form of a stationary structure was obtained as time increases or decayed to a system of waves. Numerical experiments showed that the stationary structures of specific discontinuities are stable to perturbations not only of small amplitude but also of finite amplitude. In the case when the perturbations were so large that decay of the initial wave occurred, a solution was formed consisting of a sequence of waves as time increases: a fast simple or shock wave $A \to A_p$, and a specific discontinuity $A_p \to C_1$ (the point C_1 is situated next to the point N in Fig. 7,a,b) and moves with a slower velocity than the slow simple or shock wave. The wave $A \to C_1$ in all the numerical experiments was established after the interaction with any perturbations.

The generalized problem of the decay of an arbitrary discontinuity was solved for various initial data (2.3). Below, we describe, as examples, the results of a numerical solution of a number of initial-boundary-value problems for Eq. (2.1) for the following values of the parameters

$$m = 3, \quad \mu = 0.05, \quad f = 1, \quad g = 1, \quad \kappa = 4, \quad u^r_1 = 2, \quad u^r_2 = 0.4$$

(the point A in Figs. 8–10) for different values of u^1_α and $u^0_\alpha(x)$.

Example 1. $u^l_1 = -2$, $u^l_2 = 0.4$ (the point P in Fig. 8,c) and the functions $u^0_\alpha(x)$ are a monotonic transition from the values u^1_α to the values u^r_α (the scheme in Fig. 8,a).

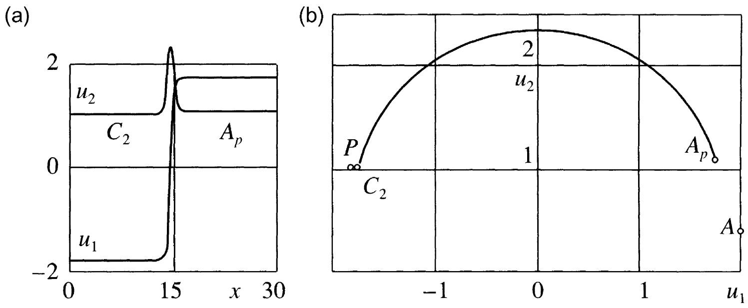

Fig. 9.

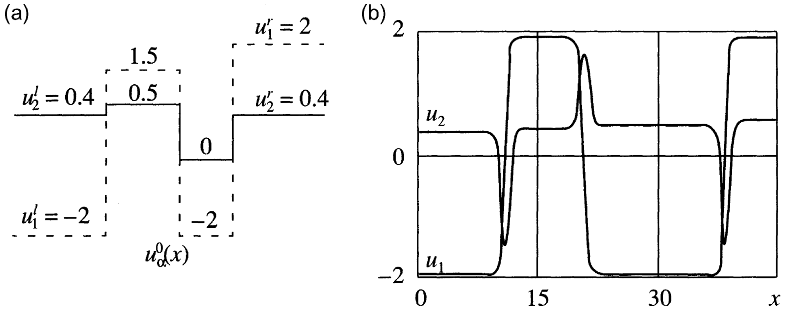

Fig. 10.

For long time (when the waves have already seperated) the solution is a sequence of waves: a fast simple wave $A \to A_p$ (Fig. 8,c), a singular discontinuity $A_p \to C_1$, and a slow shock wave $C_1 \to P$. The specific discontinuity occurring in the solution has the simple structure shown in Fig. 5. In Fig. 8 we show a graph of the specific discontinuity as a function of the x coordinate at a fixed instant of time t (Fig. 8,b) and in the (u_1, u_2) plane (Fig. 8,c).

In this example, as in the next ones, we do not show waves which propagate in front of and behind the specific discontinuity. A priori evolution discontinuities (transitions from the initial point A to points of evolution sections of the shock adiabatic) may be present in the solutions. The stationary structure of these discontinuities was investigated in Ref. 6 for a model involving only viscous terms. In this model, which takes into account the dispersion due to viscosity, these discontinuities will contain oscillations inside the structure due to dispersion.

Example 2. $u_1^l = -1.7, u_2^l = 1$ (the point P in Fig. 9,b) and the functions $u_\alpha^0(x)$ represent a monotonic transition from the values u_α^1 to the values u_α^r.

The solution, represented in Fig. 9,a,b, consists of a sequence of waves: a fast simple wave $A \to A_p$, a specific discontinuity $A_p \to C_2$, and a slow simple wave $C_2 \to P$. The structure of the specific discontinuity corresponds to the simple structure shown in Fig. 5.

Example 3. $u_1^l = -2, u_2^l = 0.4$ (the coordinates of the point P (Fig. 10) are exactly the same as in Example 1), and the functions $u_\alpha^0(x)$ are represented in Fig. 10,a.

The asymptotic form obtained represents a sequence of waves: a fast simple wave, a sequence of three specific discontinuities with a simple structure and a slow shock wave. The sequence of three specific discontinuities, having simple structures, is shown in Fig. 10,b.

An investigation of the stationary structures of the specific discontinuities showed that, for various values of the parameters, there are two “simple” structures (Fig. 5) $A \to C_1$ and $A \to C_2$. If the initial conditions of the generalized problem of the decay of an arbitrary discontinuity are specified in the form of steps (as in Examples 1 and 2), then, if for the chosen initial data there is no solution not involving specific discontinuities, the solution consists of a sequence of waves involving only simple structures of the specific discontinuities. Note that, even in the case when the right and left boundary conditions are arranged in the same coordinate quadrant and a solution is possible without specific discontinuities, the solution may involve a sequence of specific discontinuities.

In Examples 1 and 3 we took the same boundary conditions, but the functions $u_\alpha^0(x)$ were different. The solutions obtained are different and involve a different number of specific discontinuities. A comparison of the initial functions in these examples illustrates the possibility of obtaining solutions with a different number of specific discontinuities for problems with the same boundary conditions.

For certain ranges of the parameters, which specify the discontinuity, for monotonic functions $u_\alpha^0(x)$ solutions of the generalized problem of the decay of a discontinuity exist involving specific discontinuities, although solutions with permissible classical discontinuities exist simultaneously (for this it is necessary to specify the functions $u_\alpha^0(x)$ in a non-monotonic form).

Numerical experiments, carried out for other sets of parameters $m, \mu, u_\alpha^r, u_\alpha^l$, showed that the above features are maintained.

There are similar non-linear phenomena for the model described, if we keep the ratios m/μ and the initial conditions, considered as a function of $x/\sqrt{\mu}$ (or, if the boundary conditions are specified, $t/\sqrt{\mu}$), and hence the results obtained and described above can be regarded as solutions of the same equations with another set of parameters.

An investigation of the problem of the non-uniqueness of the solutions of self-similar wave problems, described by non-linear hyperbolic equations, was carried out within the framework of a model which includes a description of small-scale dispersion and dissipation processes inside the structure of the discontinuities. It has been shown on the basis of numerical experiments that all possible solutions can be realized in the region of non-uniqueness as asymptotic forms of the complete system of equations, which takes into account small-scale processes.

Asymptotic forms, involving simple stationary structures, are obtained most often. Asymptotic forms with discontinuities, having complex stationary structures, have small regions of attraction in the space of functions which specify "smoothing". An asymptotic solution with a complex structure can only arise if the specified initial functions are close to this solution.

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